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Coherent States of Two-Mode q-Oscillators With a q Root of Unity

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When q is a root of unity, the coherent states of two-mode q-oscillators are constructed by introducing the generalized Grassmann variables.

1. INTRODUCTION

The theory of quantum groups (Drinfeld, 1986; Jimbo, 1985; Woronowicz, 1987) has led to the generalization (deformation) of the oscillator (boson, fermion) algebras in several directions. The development of differential calculus in noncommutative (quantized) spaces has identified multimode systems of deformed creation and annihilation operators covariant under the actions of quantum groups (Pusz, 1989; Pusz and Woronowicz, 1989; Wess and Zumino, 1990). Generalization of the usual boson–fermion realizations to quantized Lie algebras and superalgebras have resulted in the study of single-mode deformed bosons (Biedenharn, 1989; Macfarlane, 1989) and fermions (Chaichian and Kulish, 1990; Parthasarathy and Viswanathan, 1991).

A single-mode q-oscillator with the creation ($a^\dagger$), annihilation (a), and number (N) operators obeying the relations

$$aa^\dagger - qa^\dagger a = 1, \qquad [N, a] = -a, \qquad [N, a^\dagger] = a^\dagger$$

has been the subject of study by some authors (Arik and Coon, 1976; Coon *et al.*, 1972; Kuryshkin, 1980) in the past, independent of the recent developments due to the theory of quantum groups.

When the deformation parameter q is real, the first relation of the aforementioned equation is invariant under the Hermitian conjugation. So, $a^\dagger$ can be interpreted as a Hermitian conjugate operator of a. But, the situation is different when q is a complex number.

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In this paper, we restrict our discussion to the case that q is a root of unity. In this case, the first relation of the aforementioned equation is not invariant under the Hermitian conjugation any more. Thus, we should rewrite the q-oscillator algebra as follows:

$$aa_+ - qa_+a = 1,$$

where a_+ is not a Hermitain conjugate of a. But, a and a_+ play roles of lowering and raising operators, respectively, if the following relations maintain:

$$[N, a_+] = a_+, \qquad [N, a] = -a.$$

In this paper, we discuss the two-mode q-oscillator system which is covariant under some quantum group $sl_q(2)$. In the following we restrict our discussion to the case that q is a $(s+1)$th primitive root of unity

$$q^{s+1} = 1 \quad \text{or} \quad q = e^{\frac{2\pi}{s+1}i}.$$

2. $sl_q(2)$-COVARIANT OSCILLATOR ALGEBRA

When q is real, quantum group covariant oscillator algebra was firstly introduced by Pusz and Woronowicz (1989). They demanded the $gl_q(n)$-covariance among step operators. Following their technique, we can write the $sl_q(2)$-covariant two-mode oscillator algebra as follows:

$$\begin{aligned} a_1a_2 &= qa_2a_1, \\ a_{1+}a_{2+} &= q^{-1}a_{2+}a_{1+}, \\ a_1a_{2+} &= qa_{2+}a_1, \\ a_2a_{1+} &= qa_{1+}a_2, \\ a_1a_{1+} - q^2a_{1+}a_1 &= 1, \\ a_2a_{2+} - q^2a_{2+}a_2 &= 1 + (q^2 - 1)a_{1+}a_1. \end{aligned} \tag{1}$$

When $q^{s+1} = 1$, from the algebra (1), we see that both a_i^{s+1} and a_{i+}^{s+1} commute with all operators of algebra (1), which means that they are central elements of algebra (1). So we can set

$$a_{i+}^{s+1} = a_i^{s+1} = 0 \tag{2}$$

and we have the finite-dimensional representation.

Now we will prove the $sl_q(2)$-covariance of the algebra (1) explicity. In order to do so, we should introduce the $sl_q(2)$-matrix. An $sl_q(2)$-matrix can be written in the form

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where the following commutation relations hold

$$\begin{aligned} ad - da &= \left(q - \frac{1}{q}\right)bc, \\ ab = qba, &\qquad cd = qdc, \\ ac = qca, &\qquad bd = qdb, \\ bc = cb, &\qquad \det_q M = ad - qbc = 1. \end{aligned} \tag{3}$$

By the $sl_q(2)$-covariance of the system, it is meant that the linear transformations

$$\begin{aligned} M\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} \\ (a_{1+}\ a_{2+})\,M^{-1} &= (a_{1+}\ a_{2+})\begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} = (a'_{1+}\ a'_{2+}) \end{aligned} \tag{4}$$

lead to the same commutation relations (1) for (a'_1, a'_{1+}) and (a'_2, a'_{2+}). It should be noted that the particular coupling between the two modes is completely dictated by the required $sl_q(2)$-covariance.

The Fock space representation of the algebra (1) can be easily constructed by introducing the Hermitian number operators $\{N_1, N_2\}$ obeying

$$[N_i, a_j] = -\delta_{ij}a_j, \qquad [N_i, a_{j+}] = \delta_{ij}a_{j+} \quad (i, j = 1, 2). \tag{5}$$

Let $|0, 0\rangle$ be the unique ground state of this system satisfying

$$N_i|0, 0\rangle = 0, \qquad a_i|0, 0\rangle = 0 \quad (i = 1, 2) \tag{6}$$

and $\{|n, m\rangle \mid n, m = 0, 1, 2, \ldots, s\}$ be the set of the orthogonal number eigenstates

$$N_1|n, m\rangle = n|n, m\rangle, \qquad N_2|n, m\rangle = m|n, m\rangle, \qquad \langle n, m \mid n', m'\rangle = \delta_{nn'}\delta_{mm'}. \tag{7}$$

From the algebra (1) the representation is given by

$$\begin{aligned} a_1|n, m\rangle &= \sqrt{[n]}|n-1, m\rangle, \qquad a_2|n, m\rangle = q^n\sqrt{[m]}|n, m-1\rangle, \\ a_{1+}|n, m\rangle &= \sqrt{[n+1]}|n+1, m\rangle, \qquad a_{2+}|n, m\rangle = q^n\sqrt{[m+1]}|n, m+1\rangle, \end{aligned} \tag{8}$$

where the q-number $[x]$ is defined as

$$[x] = \frac{q^{2x} - 1}{q^2 - 1}.$$

3. COHERENT STATES

The coherent state of $sl_q(2)$-covariant oscillator algebra is defined as

$$a_1|z_1, z_2\rangle = z_1|z_1, z_2\rangle,$$
$$a_2|z_1, z_2\rangle = z_2|z_1, z_2\rangle, \tag{9}$$

So, we get

$$a_i^{s+1}|z_1, z_2\rangle = z_i^{s+1}|z_1, z_2\rangle.$$

We know that $|z_1, z_2\rangle \neq 0$ and $a_i^{s+1} = 0$, so we have

$$z_1^{s+1} = z_2^{s+1} = 0. \tag{10}$$

These are generalized Grassmann variables (or $(s+1)$th nilpotent variables). In an analogous way to an ordinary fermion system, we can assume that

$$z_1|n, m\rangle = q^n|n, m\rangle z_1,$$
$$z_2|n, m\rangle = q^m|n, m\rangle z_2. \tag{11}$$

From the fact that $\langle n, m \mid n, m\rangle = 1$ and $[z_i, \langle n, m \mid n, m\rangle] = 0$, we have

$$z_1\langle n, m| = q^{-n}\langle n, m|z_1,$$
$$z_2\langle n, m| = q^{-m}\langle n, m|z_2. \tag{12}$$

Using the above commutation relations between number eigenstates and coherent variables (generalized Grassmann variables), we see that

$$a_1z_1 = qz_1a_1,$$
$$a_2z_2 = qz_2a_2,$$
$$a_1z_2 = z_2a_1,$$
$$a_2z_1 = z_1a_2, \tag{13}$$

$$a_{1+}z_1 = q^{-1}z_1a_{1+},$$
$$a_{2+}z_2 = q^{-1}z_2a_{2+},$$
$$a_{2+}z_1 = z_1a_{2+},$$
$$a_{1+}z_2 = z_2a_{1+}. \tag{14}$$

From the $sl_q(2)$-covariant oscillator algebra, the coherent variables z_i's and their complex conjugate variables z_i^*'s satisfy the following commutation relations:

$$\begin{aligned} z_1 z_2 &= q^{-1} z_2 z_1, \\ z_1^* z_2^* &= q^{-1} z_2^* z_1^*, \\ z_1 z_2^* &= q z_2^* z_1, \\ z_2 z_1^* &= q^{-1} z_1^* z_2, \\ z_i z_i^* &= z_i^* z_i = |z_i|^2. \end{aligned} \tag{15}$$

Then, $|z_i|^2$ (hermitian norm of z_i) commute with all coherent variables, that is to say,

$$[|z_1|^2, z_2] = [|z_2|^2, z_1] = [|z_1|^2, |z_2|^2] = 0.$$

If we define the generalized Grassmann integral as

$$\int dz_2^*\, dz_2\, dz_1^*\, dz_1\, (z_2^*)^n z_2^m (z_1^*)^p z_1^l = \delta_{ns}\delta_{ms}, \delta_{ps}\delta_{ls}$$

then there exist $(s+1)^2$ coherent states

$$\begin{aligned} |z_1, z_2\rangle_{nm} = \sqrt{[s-n]![s-m]!} \sum_{r=0}^{s-n} \sum_{p=0}^{s-m} \left(q^{r(r+2n+1)+p(p+2m+1)}\right)^{-1/2} \\ \times z_1^{r+n} z_2^{p+m} |rp\rangle, \end{aligned} \tag{16}$$

where $n, m = 0, 1, 2, \ldots, s$.

Then, all coherent states $\{|z_1, z_2\rangle_{nm} \mid n, m = 0, 1, 2, \ldots, s\}$ constitute the complete set and they obey

$$\int dz_2^*\, dz_2\, dz_1^*\, dz_1\, {}_{nm}\langle z_1, z_2 \mid z_1, z_2\rangle_{n'm'} = \delta_{nn'}\delta_{mm'} \tag{17}$$

or

$$\sum_{n,m=0}^{s} \int dz_2^*\, dz_2\, dz_1^*\, dz_1\, |z_1, z_2\rangle_{nm}\, {}_{nm}\langle z_1, z_2| = 1. \tag{18}$$

4. CONCLUSION

In this paper, we have proposed a $sl_q(2)$-covariant oscillator algebra when q is a $(s+1)$th primitive root of unity and studied its representation and some basic characteristics. Using the idea of the generalized Grassmann calculus, we obtained a lot of coherent states of $sl_q(2)$-covariant oscillator algebra with a q root of unity. The results discussed in this paper can be easily extended to a more general case, $sl_q(n)$-covariant multimode oscillator algebra.

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